MATH3401

Q1 a) $xy'' - (1+x)y' + y = 0$

If $y = e^x$: $\cancel{x}e^x - (\cancel{1}+\cancel{x})e^x + \cancel{e^x} = 0$ ✓

Try $y = ue^x$ in $xy'' - (1+x)y' + y = x^2$

$$\Rightarrow \quad x(u''+2u'+u)e^x - (1+x)(u'+u)e^x + ue^x = x^2$$

$$\Rightarrow \quad u''e^x(x) + u'e^x(2x - x - 1) + ue^x(\underbrace{x-1-x+1}_{0}) = x^2$$

If $u' = v$ then $v' + v\left(1 - \frac{1}{x}\right) = xe^{-x}$

IF is $\frac{e^x}{x}$ $\Rightarrow$ $\left[\frac{ve^x}{x}\right]' = 1 \Rightarrow v = u' = x^2e^{-x} + Bxe^{-x}$

$$\Rightarrow \quad u = A + B\int^x te^{-t}\,dt + \int^x t^2e^{-t}\,dt$$

$$= A + B\left\{\left[-te^{-t}\right]^x + \int^x e^{-t}\,dt\right\} + \left\{\left[-t^2e^{-t}\right]^x + \int^x 2te^{-t}\,dt\right\}$$

$$= A + B\left\{-xe^{-x} - e^{-x}\right\} + \left\{-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right\}$$

$y = e^x u$ $\quad B \to -B$ & $\quad y = Ae^x + B(x+1) - (x^2+2x+2)$

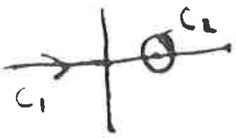
b) $xy'' - (1+x)y' - y = 0$. If $y = \int_c e^{xt}f(t)\,dt$ then

substitution $\Rightarrow \int_c xe^{xt}f(t^2-t) + -(t+1)fe^{xt}\,dt$

$$\Rightarrow \left[e^{xt}f\,t(t-1)\right]_c - \int_c e^{xt}\left[f't(t-1) + f(2t-1) + (t+1)f\right]dt = 0$$

$$\Rightarrow \left[e^{xt}f(t)(t-1)\right]_c = 0 \quad \& \quad f't(t-1) = -3tf \Rightarrow f = \frac{A}{(t-1)^3}$$

so $y = \int_c \frac{e^{xt}}{(t-1)^3}\,dt$ with c chosen so $\left[e^{xt}\frac{t}{(t-1)^2}\right]_c = 0$



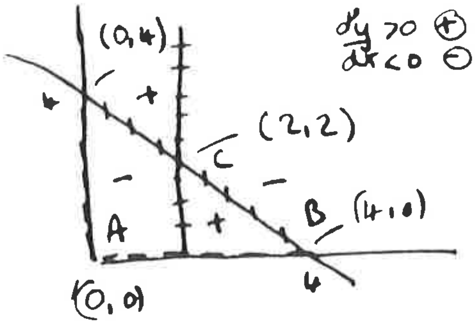
C_1 generates $y = \int_{-\infty}^{0} \frac{e^{xt}}{(t-1)^3}\,dt = -\int_0^{\infty} \frac{e^{-xt}}{(t+1)^3}$

C_2 generates $\oint_{C_2} \frac{e^{xt}}{(t-1)^3}\,dt \propto \frac{d^2}{dt^2}e^{xt}\Big|_{t=1} = x^2e^x$

2) a) $\dot{x} = 4x - x^2 - xy$, $\dot{y} = -2y + xy$

Vertical nullclines : $\dot{x} = 0 \Rightarrow x = 0$ or $x + y = 4$

Horizontal " : $\dot{y} = 0 \Rightarrow y = 0$ or $x = 2$



$\frac{dy}{dx} > 0$ ⊕
$\frac{dy}{dx} < 0$ ⊖

A vertical & horizontal nullcline cross at

$A = (0,0)$, $B = (4,0)$, $C = (2,2)$

These are the critical points

The Jacobian matrix is $\begin{pmatrix} 4 - 2x - y & -x \\ y & -2 + x \end{pmatrix}$

At A this is $\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$, two eigenvalues, different sign $\Rightarrow$ saddle

B this is $\begin{pmatrix} -4 & -4 \\ 0 & 2 \end{pmatrix}$, two eigenvalues different sign $\Rightarrow$ saddle

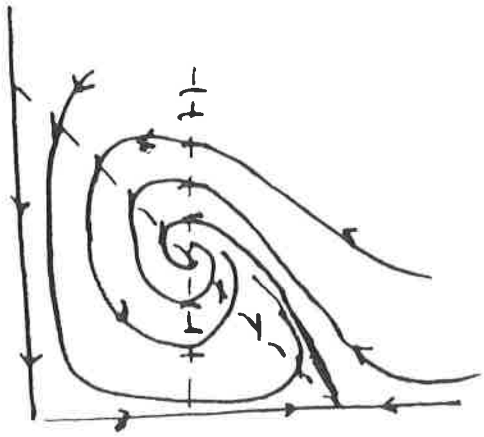
locally $\frac{dY}{dX} = \frac{2Y}{-4X - 4Y}$ & Straight line trajectories, $y = mx$ gives

$m(m+1) = -\frac{1}{2}m \Rightarrow m = 0$ or $m = -3/2$.

C, this is $\begin{pmatrix} -2 & -2 \\ 2 & 0 \end{pmatrix}$, eigenvalues satisfy $(-2-\lambda)(-\lambda) + 4 = 0$

$\Rightarrow$, $\lambda = -1 \pm \sqrt{3}i$

spiral point, stable as $Re(\lambda) < 0$



2b) $\underline{r\dot{r} = x\dot{x} + y\dot{y}}$ $\qquad$ $\underline{r^2\dot{\theta} = x\dot{y} - y\dot{x}}$

$\dot{x} = x(x^2+y^2-2x-3) - y = x(r^2-2r\cos\theta-3) - y$

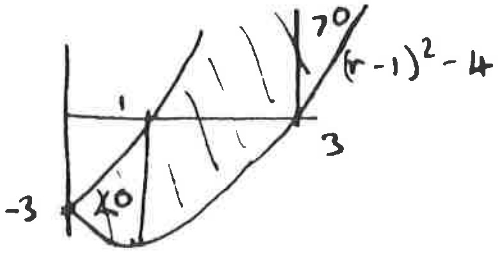
$\dot{y} = y(x^2+y^2-2x-3) + x = y(r^2-2r\cos\theta-3) + x$

$\Rightarrow r\dot{r} = (x^2+y^2)(r^2-2r\cos\theta-3) - xy + yx \quad \Rightarrow \dot{r} = r(r^2-2r\cos\theta-3$

& $r^2\dot{\theta} = xy(\;) + x^2 - xy(\;) + y^2 = r^2 \Rightarrow \dot{\theta} = 1$

Now $\quad r^2-2r-3 \leq r^2-2r\cos\theta-3 \leq r^2+2r-3$

$\Rightarrow (r-1)^2-4 \leq r^2-2r\cos\theta-3 \leq (r+1)^2-4$



For $r<1$, $\dot{r}<0$

For $r>3$ $\dot{r}>0$

& as $\dot{\theta} \neq 0$ & in $1<r<3$ & so

there are no critical points PB $\Rightarrow$ limit cycle

in $1<r<3$

3 a) $\ddot{x} + \varepsilon \dot{x} f(x) + x = 0$

$x = a\cos\theta + \varepsilon x_1(\theta) + \dots \qquad \theta = t(1+\varepsilon\omega_1 \dots)$

$\Rightarrow (1+\varepsilon\omega_1+\dots)^2(-a\cos\theta + \varepsilon\ddot{x}_1 + \dots) + \varepsilon(1+\dots)(-a\sin\theta+\dots)f(a\cos\theta)$
$+ a\cos\theta + x_1 = 0$

$\Rightarrow \ddot{x}_1 + x_1 = 2a\omega_1\cos\theta + a\sin\theta f(a\cos\theta)$

To avoid secular terms, set the coefficients of $\sin\theta$ & $\cos\theta$ in the Fourier series of rhs to zero:

$$2a\omega_1 \underbrace{\int_{-\pi}^{\pi}\cos^2\theta\,d\theta}_{\neq 0} + a\underbrace{\int_{-\pi}^{\pi}\cos\theta\sin\theta f(a\cos\theta)\,d\theta}_{=0} = 0 \quad \Rightarrow \underline{\omega_1 = 0}$$

$$2a\omega_1\int_{-\pi}^{\pi}\sin\theta\cos\theta\,d\theta + a\int_{-\pi}^{\pi}\sin\theta\sin\theta f(a\cos\theta)\,d\theta = 0$$

$$\Rightarrow \underline{\int_{-\pi}^{\pi}\sin^2\theta f(a\cos\theta)\,d\theta = 0}$$

b) If $f(x) = x^2 - \alpha^2$ this gives

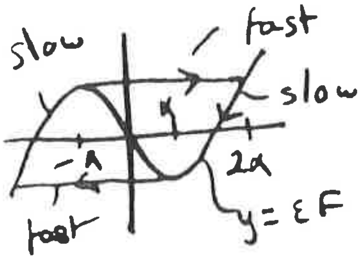
$$\int_{-\pi}^{\pi} a^2\cos^2\theta\sin^2\theta\,d\theta = \int_{-\pi}^{\pi}\sin^2\theta \cdot \alpha^2\,d\theta$$

$$\Rightarrow a^2 \cdot 2\pi \cdot \tfrac{1}{4}\cdot\tfrac{1}{2} = \alpha^2 \cdot 2\pi \cdot \tfrac{1}{2} \quad \Rightarrow \underline{a = 2\alpha}$$

c) [VdP eqn ($\alpha = 1$) seen & others]

$y = \dot{x} + \varepsilon F(x)$, $F'(x) = f(x) \Rightarrow \dot{y} = \ddot{x} + \dot{x}F' = \ddot{x} + \dot{x}f(x) = -x$

So $\dot{x} = y - \varepsilon F$, $\dot{y} = -x$. Closed trajectory is as shown.



If $\varepsilon \gg 1$ the period is dominated by the slow parts of the orbit, as shown. $F' = 0$ at $x = \pm\alpha$

$F(-\alpha) = -\frac{1}{3}\alpha^3 + \alpha^2\alpha = 2\alpha^3/3$

$F(x) = F(-\alpha) \Rightarrow \frac{1}{3}x^3 - \alpha^2 x = 2\alpha^3/3 \Rightarrow x^3 - 3\alpha^2 x - 2\alpha^3 = 0$

$\Rightarrow (x+\alpha)^2(x-2\alpha) = 0$

$$T = 2\int dt = 2\int_{2\alpha}^{\alpha}\frac{dx}{\dot{x}} = 2\int_{2\alpha}^{\alpha}\frac{dy}{dx}\frac{dx}{dy/dt} = 2\int_{2\alpha}^{\alpha}\frac{\varepsilon(x^2-\alpha^2)}{-x}\,dx = 2\int_{\alpha}^{2\alpha} x - \frac{\alpha^2}{x}\,dx$$

$$= 2\varepsilon\left[\tfrac{1}{2}3\alpha^2 - \alpha^2\ln 2\right] = \alpha^2\varepsilon(3 - 2\ln 2)$$

4) $\ddot{x} + x = \varepsilon f(x, \dot{x})$

with $T = \varepsilon t$, $\frac{d}{dt} \to \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}$ & with $x = x_0(t,T) + \varepsilon x_1(t,T) \ldots$

$$\left(\frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial T \partial t} + \ldots\right)(x_0 + \varepsilon x_1 + \ldots) + (x_0 + \varepsilon x_1 \ldots) = \varepsilon f(x_0, x_{0t}) + \ldots$$

$$\Rightarrow \quad x_{0tt} + x_0 = 0 \quad \Rightarrow \quad x_0(t,T) = A(T)\sin(\underbrace{t + \varphi(T)}_{\chi})$$

$$\& \quad x_{1tt} + x_1 = f(A\sin\chi, A\cos\chi) - 2\underbrace{\frac{\partial}{\partial T}(A(T)\cos\chi)}_{A_T\cos\chi - \varphi_T A\sin\chi}$$

Remove possibility of secular terms by setting Fourier coefficients of $\sin\chi$ & $\cos\chi \equiv 0$

$$\Rightarrow \int_{-\pi}^{\pi} \cos\chi\, f(A\sin\chi, A\cos\chi)\,d\chi = 2\frac{\partial A}{\partial T}\underbrace{\int_{-\pi}^{\pi}\cos^2\chi\,d\chi}_{\pi} - 2A\frac{\partial\varphi}{\partial T}\underbrace{\int_{-\pi}^{\pi}\cos\chi\sin\chi\,d\chi}_{0}$$

$$\Rightarrow \frac{\partial A}{\partial T} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\chi\, f(A\sin\chi, A\cos\chi)\,d\chi$$

$$\& \quad \int_{-\pi}^{\pi}\sin\chi\, f(A\sin\chi, A\cos\chi)\,d\chi = 2\frac{\partial A}{\partial T}\int_{-\pi}^{\pi}\sin\chi\cos\chi\,d\chi - 2A\frac{\partial\varphi}{\partial T}\int_{-\pi}^{\pi}\sin^2\chi\,d\chi$$

$$\Rightarrow A\frac{\partial\varphi}{\partial T} = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin\chi\, f(A\sin\chi, A\cos\chi)\,d\chi$$

i) If $f(x,\dot{x}) = x - \dot{x}$ then

$$\frac{\partial A}{\partial T} = \frac{1}{2\pi}\int_{-\pi}^{\pi} A\cos\chi\sin\chi - A\cos^2\chi\,d\chi = \frac{1}{2\pi}\left(-2\pi\cdot\frac{1}{2}\right)A = -A/2$$

$$\Rightarrow A = A(0)\,e^{-T/2}$$

$$A\frac{\partial\varphi}{\partial T} = -\frac{1}{2\pi}\int_{-\pi}^{\pi} A\sin^2\chi - A\sin\chi\cos\chi\,d\chi = -\frac{1}{2\pi}\cdot\left(2\pi\cdot\frac{1}{2}\right)A$$

$$\Rightarrow \frac{\partial\varphi}{\partial T} = -\frac{1}{2} \quad \& \quad \varphi = \varphi(0) - \frac{1}{2}T$$

$$x(t) = A(0)\,e^{-\varepsilon t/2}\sin\left(t + \varphi(0) - \frac{1}{2}\varepsilon t\right)$$

BC $\Rightarrow$ $A(0) = 0$
$\varphi(0) = \pi/2$.

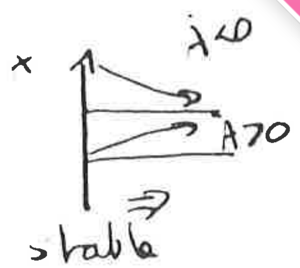
~~b)~~

b) $f(x,\dot{x}) = -\dot{x}(\dot{x}-\alpha)(\dot{x}-\beta)$

$$\frac{\partial A}{\partial T} = \frac{1}{2\pi}\int_{-\pi}^{\pi} -A\cos^2\chi\left(A^2\cos^2\chi - (\alpha+\beta)A\cos\chi + \alpha\beta\right)d\chi$$

$$= \frac{1}{2\pi}\left(-A^3\right)\frac{3\pi}{4} + A^2\cdot 0 + -\alpha\beta A\cdot\frac{2\pi}{2\pi}\cdot\frac{1}{2}$$

$$= -\frac{\alpha\beta}{2}A - \frac{3}{8}A^3 = \frac{A}{2}\left(|\alpha\beta| - \tfrac{3}{4}A^2\right)$$



$$\frac{\partial A}{\partial T} = 0 \Rightarrow A = 2\sqrt{\frac{|\alpha\beta|}{3}}$$

$$A\frac{\partial\varphi}{\partial T} = -\frac{1}{2\pi}\int_{-\pi}^{\pi} -A\sin\chi\,\mathrm{fun}(\cos\chi)\,d\chi = 0. \Rightarrow \varphi = \text{const.}$$

$= 0$ wlog for periodic solns

$$x(t) = 2\sqrt{\frac{|\alpha\beta|}{3}}\sin(t)$$

5 a i) $I(x) = \int_0^T e^{-xt} f(t)\,dt$ & $f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^n$

$\Rightarrow$ $I(x) \sim \sum_{n=0}^{\infty} \frac{a_n (\alpha+n)!}{x^{n+\alpha+1}}$

ii) $K_0(x) = \int_1^{\infty} \frac{e^{-xt}}{\sqrt{t^2-1}}\,dt = \int_0^{\infty} \frac{e^{-xu} e^{-x}}{\sqrt{u^2+2u}}\,du = e^{-x} \int_0^{\infty} \frac{e^{-xu}\,du}{\sqrt{2}\sqrt{u}\,(1+u/2)^{1/2}}$

$t = u+1$

& as $(1+u/2)^{-1/2} \sim 1 - u/4 \cdots$ as $u \to 0$, Watsons Lemma now gives, $\alpha = -1/2$ in above

$$K_0(x) \sim \frac{e^{-x}}{\sqrt{2}} \frac{1}{x^{1-1/2}} \left((-1/2)! - \frac{(1/2)!}{4x} \cdots \right)$$

$(-1/2)! = \sqrt{\pi}$ & $(1/2)! = 1/2(-1/2)! = \sqrt{\pi}/2$

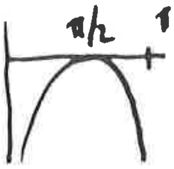
$$K_0(x) \sim e^{-x} \sqrt{\frac{\pi}{2x}} \left[1 - \frac{1}{8x} \cdots \right]$$

iii) $I(x) = \int_0^{\pi/4} \sqrt{\sin t}\, e^{-x\sin^4 t}\,dt$. Put $u = \sin^4 t$
$du = 4\sin^3 t \cos t\,dt$

$$= \int_0^1 \frac{u^{1/8} e^{-xu}\,du}{4u^{3/4}\sqrt{1-\sqrt{u}}} = \frac{1}{4}\int_0^1 u^{-5/8} \frac{e^{-xu}}{\sqrt{1-\sqrt{u}}}\,du \qquad (\alpha = -5/8)$$

& as $\sqrt{1-\sqrt{u}} \sim 1$ as $u \to 0$ $I(x) \sim \frac{1}{4} \frac{(-5/8)!}{x^{3/8}}$

b) $F(x,\nu) = \int_0^{\pi} \cos(x\cos\theta)\sin^{2\nu}\theta\,d\theta = \int_0^{\pi} \cos(x\cos\theta)\, e^{2\nu\ln\sin\theta}\,d\theta$



$\varphi(\theta) = \ln\sin\theta$, $\varphi'(\theta) = \frac{\cos\theta}{\sin\theta}$, φ has max at $\theta = \pi/2$

where $\varphi''(\theta) = -1 - \cos^2\theta/\sin^2\theta = -1$

Laplace's method gives $F \sim \sqrt{\frac{2\pi}{\nu|-1|}}\, \overbrace{\cos(x\cos\pi/2)}^{\cos 0 = 1}\, e^{2\nu \overbrace{\ln\sin\pi/2}^{\ln 1 = 0}}$

$\nu \to \infty$

$\int e^{\varphi(t)x} f(t)\,dt$

$\sim \sqrt{\frac{2\pi}{x|\varphi''(t_0)|}}\, f(t_0)\, e^{x\varphi(t_0)}$ $\qquad = \sqrt{\frac{2\pi}{\nu}}$

$x \to \infty$ $\varphi'(t_0) = 0$
$\varphi''(t_0) < 0$